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CONJUGATION OF MORSE FUNCTION ON 3-MANIFOLDS

The questions when two Morse function on closed manifolds are conjugated is investigated. Using the handle decompositions of manifolds the condition of conjugation is formulated.

For each Morse function on 3-manifold the ordered generalized Heegaard diagram is built. The criteria of Morse function conjugation are given in the terms of equivalence of such diagrams.

Let M is a smooth manifold, f and g is Morse function on it. Functions f and g are called conjugated, if there exist homeomorphisms $h : M \to M$, $h' : R^1 \to R^1$ such, that $f\ h' = gh$ and homeomorphism h' preserve the orientation.

If we demand that conjugated homeomorphisms are isotopic to identity diffeomorphism then in the case of 1-connected n-manifolds with $n > 5$ there us a criterion of Morse function conjugate in the terms of ordering basic chain complex equivalence [1].

Collar handle decomposition.

Collar handle decomposition is a sequence of imbedding $M_0 \subset M_0' \subset M_1 \subset M_1' \subset M_2 \subset ... \subset M_N = M$ such that M_0 is a union of n-disks (0-handels). M_i' is obtained from M_i by gluing collar $N_i \times [0,1]$, where $N_i = \partial\ M_i$, and M_{i+1} is obtained from M_i' by gluing handles. We fix the projection $\pi : N_i \times [0,1] \to [0,1]$ on each collar.

Collar handle decomposition is isomorphic, if there exist homeomorphism between manifolds which maps handles on handles, collars on collars and commute with projections π.

Having Morse function $f : M \to R^1$ with critical values $\{1, 2, ..., N\}$ let us build collar handle decomposition in a such way that interiors of collars $N_i \times [0,1]$ will be gomeomorphic in fibers to connected component of $M \backslash f^{-1}(\{1, 2, ..., N\})$.

The first way to do this:

Let p be a critical point. Let us consider such it chart that $p = (0, 0, ..., 0)$ and

$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

We denote by D^k, D^{n-k} the disks with enough small radius ε :

$$D^k = \{(x_1, x_2, ..., x_k, 0, 0, ..., 0) : \sum_{i=1}^{k} x_i^2 \le \varepsilon\}$$

$$D^n{-}^k = \{(0, 0, ..., 0, x_{k+1}, ..., x_n) : \sum_{i=k+1}^{n} x_i^2 \le \varepsilon\}.$$

These disks are core and cocore of handle

$$H_j = D^k \times D^n{-}^k = \{(x_1, x_2, ..., x_n) : \sum_{i=1}^{k} x_i^2 \le \varepsilon, \sum_{i=k+1}^{n} x_i^2 \le \varepsilon\}.$$

Cut manifold M by critical levels and such constructed handles for each critical point. We denote by W_i the closure of connected components of this set.

Lemma 1. W_i is gomeomorphic to $N_i \times [0,1]$, where N_i is a regular level in W_i.

Proof. Let us fix Reiman metric on the manifold M and construct vector field grad f that is transversal to boundaries W_i. Because N_i is a transversal section of grad f then W_i is gomeomorphic to $N_i \times [0,1]$.

Consecutive gluing of handles H_j and collars $N_i \times [0,1]$ we obtain collar handle decomposition.

Conversely having collar handle decomposition by smoothing corners we can construct Morse function, which induce given handle decomposition.

Second way:

Cutting manifold M by critical level we obtain the interiors $N_i \times (0,1)$ of collar $N_i \times [0,1]$. Manifold M can be obtained under identification of corresponding points from boundaries of neighboring collars $N_i \times [0,1]$

and $N_{i+1} \times [0,1]$ and contraction of spheres S^{k-1} and S^{n-k-1} from $N_i \times \{1\}$ and $N_{i+1} \times \{0\}$, correspondingly, to critical k-point.

Indeed, if we fix Reiman metric and build vector field grad f then corresponding points from the collars boundaries is the ends of cutting trajectories parts. The spheres are limits sets of trajectories that correspond to ones from stable and unstable manifolds of the critical points.

The handle decomposition can be obtained in such way: the points from neighboring collars, besides points from regular neighborhood of above spheres, are identified as previously and we glue k-handles to regular neighborhood of spheres S^{k-1} and S^{n-k-1}.

Reverse procedure is contraction of handles to critical points.

Lemma 2. Functions are conjugate if and only if correspondent collar handle decomposition is isomorphic.

Proof. If there exist conjugating homeomorphism between manifolds then if we fix Reimann metric on one manifold we can construct induce metric on another. Then under this homeomorphism integral trajectories of one field will be mapped in the integral trajectories of another. Thus from construction it is followed that correspondent collar handle decompositions are isomorphic.

Let us see that collar handle decompositions don't depend from metric chooses. Because of all metric on manifold are homotopic then the spheres S^{k-1} and S^{n-k-1}, which are constructed using its, are isotopic. Then these isotopies give fiber maps of collars. Extending these maps on handles we obtain needed homeomorphism.

Reverse, homeomorphism between manifolds, which has collar handle decompositions, gives conjugated homeomorphism between Morse functions.

It is followed from lemmas 1 and 2 that collar handle decompositions constructed in a first and second way are isomorphic.

Criteria of Morse functions conjugation.

Let us choose the such structure of direct product $N_i \times [0,1]$ on each collar that for each $t \in (0,1) : N_i \times \{t\} = f^{-1}(y)$ for appropriate regular value y. Then after removing collars from manifold and identification $N_i \times \{0\}$ with $N_i \times \{1\}$ we obtain handle decomposition without collars. This decomposition depend from structure of direct product on each collar.

Let us see that different structure of direct product correspond handles gluing on isotopic embeddings, *i.e.* diffeomorphisms p, $q : N_i \times \{0\} \to N_i \times \{1\}$ which are projection of one collar boundary component on another for correspondent structures of direct product are isotopic. Indeed, let p_t, $q_t : N_i \times [0,1] \to N_i \times \{t\}$ is projections of collar to level $N_i \times \{t\} = f^{-1}(y_t)$. Then isotopy can be given by formula

$$F(x,t) = p_1(q_t(x)).$$

Thus the next lemma is true:

Lemma 3. Two Morse function are conjugate if and only if in the handle decomposition which associated with it correspondent handles are gluing on isotopic embeddings.

Using isotopies of attaching spheres one can made handle decomposition such that attaching and belt spheres has transversal intersections and such that each handle attach to the union of less dimensional handles. We call such handle decomposition simple.

The handle decomposition is called ordered if the map of handles set to set $\{1, 2,..., N\}$ is given. Each Morse function assign order in the corresponding handle decomposition: we can choose such $h' : R^1 \to R^1$ that critical values set maps on the set of numbers $\{1, 2,..., N\}$ we put in correspondence to each handles the number of correspondent critical points number.

Ordered simple handle decompositions (OSHD) are isomorphic if there is a homeomorphism of the manifolds, which maps handles on handles, cores on cores, cocores on cocores and preserves order of handles. Denote by M^k the union of handles which indexes are no more then k and $L^k = \partial\, M^k$.

Theorem 1. Two Morse functions are conjugate if and only if from first function OSHD we can obtain second function OSHD using

1) attaching spheres isotopies in L^k with support in the boundary of handles with less numbers;

2) replacement handle H_i by $H_i \# H_j$, if number of H_i is more than the number of H_j (handles have the same index). The new handle $H_i \# H_j$ have the same number as the handle H_i.

Proof. Let us consider the isotopy of attaching sphere embeddings as in lemma 3. We transform such isotopy to general position with belt spheres of less numbers. Then under such isotopy attaching sphere of each handle don't intersect belt spheres of greatest index for arbitrary parameter of isotopy t.

If under isotopy of attaching sphere of handle H_i it pass belt sphere of handle H_j with the same index (if there is t when these spheres have intersection) then handle H_i is replaced by sum $H_i \# H_j$.

Because such isotopy has support in the boundary of the union of handles that are already attached, *i.e.* handles with the less numbers, then we obtain the condition of theorem.

Generalized Heegaard diagrams

Let $N \cup N' = M$ be a Heegaard decomposition of the manifold M, $F = \partial\ N = \partial\ N'$ be the common surface of N and N'[2]. The set $u = \{u_1,\ u_2, ...,\ u_n\}$ of non intersected closed curves on surface F are called generalized meridian system for N, if it is the boundary of disks $D_i \subset H$ and if we cut H along it we obtain disconnected union of 3-disks. Let $v = \{v_1,\ v_2, ...,\ v_m\}$ be the generalized meridian system for N'.

The triple $(F,\ u,\ v)$ is called the generalized Heegaard diagram of manifold M. Diagrams $(F,\ u,\ v)$ and $(F',\ u',\ v')$ are called gomeomorphic if there exist such homeomorphism $h : F \to F$, that $h(u) = u'$, $h(v) = v'$. Diagrams $(F,\ u,\ v)$ and $(F',\ u',\ v')$ are called semiisotopic, if there exist such isotopies $\varphi_t, \psi_t : F \to F$, that $\varphi_0 = \psi_0 = 1$, $\varphi_1(u) = u'$, $\psi_1(v) = v'$.

Let us define the operation of meridian addition: the sum $u_1 \# u_2$ of two meridian u_1 and u_2 along simple curve α, which connect u_1 and u_2 is a such component of the union neighborhood boundary $\partial U(u_1 \cup u_2 \cup \alpha)$ which don't isotopic neither u_1 not u_2.

Denote by $U_1,\ U_2, ...,\ U_k$ that domains which are obtained after cutting surface F along meridians u_1, $u_2, ...,\ u_n$ and by $V_1,\ V_2, ...,\ V_l$ — correspondent domains for meridians $v_1,\ v_2, ...,\ v_m$. Diagram is called ordered if the map σ of set $\{U_1, U_2, ...,\ U_k,\ u_1,\ u_2, ...,\ u_n,\ v_1,\ v_2, ...,\ v_m,\ V_1,\ V_2, ...,\ V_l\}$ on set $\{1,\ 2, ...,\ N\}$ is given.

Ordered generalized Heegaard diagrams (OGHD) are called equivalent if on from another can be obtained using homeomorphisms, semiisotopies of diagrams (finger moves between u_i and v_j, if $\sigma(u_i) > \sigma(v_j)$), replacement meridians u_i on $u_i \# u_j$ if $\sigma(u_i) < \sigma(u_j)$ and replacement meridians v_i on $v_i \# v_j$ if $\sigma(v_i) > \sigma(v_j)$. If we replace u_i on $u_i \# u_j$ then $\sigma(u_i \# u_j) = \sigma(u_i)$.

Morse function conjugation on 3-manifolds.

On 3-manifolds isotopies in 1) of theorem 1 in L^1 can be realized using finger moves or reverse moves between belt spheres of the handle H^1 and attaching sphere of the handle H^2 in L^1 under condition that number of the handle H^1 is less then number of H^2.

Indeed. Let functions are conjugate and number of the handle H^1 are less than number of H^2. Then when we glue handle H^2 belt sphere of handle H^1 and attaching sphere of handle H^2 can have the intersections. The isotopy of attaching sphere of the handle H^2 with respect to belt sphere of handle H^1 will be realized by finger moves and there reverse. In addition the pair of intersection points between these spheres (closed curves) will be appear or disappear.

Theorem 2. Two Morse function on 3-manifolds are conjugate if and only if associated ordered generalized Heegaard diagrams are equivalent.

Proof. Indeed, two sets of meridians from generalized Heegaard diagram are belt spheres of 1-handles and attaching spheres of 2-handles in L^1. The isotopies from 1) of theorem 1 in L^1 are realized by finger moves of meridians. 1- and 2-handles additions on OGHD correspond to addition of meridians.

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